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Comparison of two nested row-column designs containing a control

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SUMMARY

Consider nested row-column designs in which each square (termed a superblock) has 2 rows and 2 columns. Let an intersection of a row and column be a plot, and assume that only one treatment is allocated per plot. If two of the entries per superblock are filled with a control treatment (treatment 0), such that the control treatment occurs once per row and once per column, then there are a number of ways that the remaining plots may be filled.

This paper examines two different designs which are non-binary, but which have the same eigenvectors as basic contrasts. The treatment versus control contrasts are the contrasts of concern. The two designs are compared for fixed effect analysis, then for recovering information from the different strata. It is shown that when plot, row and column information are recovered and combined, the design providing lower variance on the control vs. treatment contrasts is dependent on the plot, row and column stratum variances.

KEY WORDS: nested row-column designs, canonical efficiency factors, controls, combining information

1. Introduction

This paper examines small blocks with nested rows and columns, containing a control and a number of test treatments. When blocks have size 2 and there are a number of test treatments and a control, Cox (1958) recommends assigning the control to one plot in each block and a test treatment to the remaining plot. There are natural situations in experiments where the block size is two: for example pairs of twins (Cox, 1958), paired organs e.g. kidneys (Cox, 1958), pairs of ears in an experiment comparing ear-tags for cows, and the two halves of a leaf (Samuel and Bald, 1933).

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Majumdar and Notz (1983) give a theorem for A-optimality in Incomplete Block Designs for comparing treatments with a control. They state that A-optimality translates into the minimum of $\sum_{i=1}^{v} \mathrm{Var}(\tau_0 - \tau_i)$ using the notation of the present paper. Plant breeding experiments often use control treatments.

Following on from the work by Cox (1958), Kachlicka and Mejza (1995) examine two row-column designs with split plots, with either one whole plot containing the control, or the subplot treatment being the control. They give the efficiency factors of these designs. The present paper examines a simpler structure, based on Kachlicka and Mejza's first design.

The structure considered here consists of b blocks of 4 plots each, where the plots in each block are arranged in a 2×2 row-column array. Kachlicka and Mejza (1995) describe an experiment with this structure for comparing three levels of irrigation. This structure is also discussed by Singh and Dey (1979), Ipinyomi and John (1985) and Bagchi et al. (1990).

Goodchild (1971) details two experiments carried out using trees as blocks. The first experiment looks at the effect of light micro-climate on fruit colour and apple size. There are three treatments including a control and blocks of size 2. In the second experiment unknown fruit thinning compounds are compared with controls. Here four trees are used as blocks, with a block size of 3. When the block size is 4, this naturally extends to a 2×2 design, since trees can be divided into quarters by the points of the compass. Spray treatments can then be applied to quarters.

In variety trials and plant breeding experiments, heterogeneity in two dimensions can be accounted for by laying out b blocks each in a $p \times q$ row-column array; see Singh and Dey (1979), Srivastava (1981), Patterson and Robinson (1989) and John and Williams (1995). Although rows and columns in such experiments usually contain more than 2 plots, the results given in the present paper may be a pointer to what happens more generally.

Bagchi et al. (1990) showed that if a nested row-column design satisfies the following conditions then it is universally optimal:

(i) the number of times that treatment i appears in row j of a superblock l is the same for each $j = 1, \ldots, p$ (Bagchi et al., 1990, label p as the number of rows within a block),

(ii) columns form a balanced block design.

Bagchi et al. (1990) call such designs BN-RC (Balanced Nested Row and Column designs which are non-binary in blocks and the rows form blocks of a balanced block design). Morgan and Uddin (1993) note that conditions (i) and (ii) give a design with completely symmetric information matrix (**C**-matrix), and maximum trace. Keifer (1975) points out that the two previous points are sufficient for universal optimality. Morgan and Uddin (1993) have also emphasized requirements (i) and (ii) for universal optimality, calling designs with these properties BNRCs (Balanced Nested Row-Column Designs which are non-binary in blocks and possibly non-binary in rows). Chang and Notz (1994) also showed the same conditions to give universally optimal nested row-column designs.

Bailey (1993), Morgan and Uddin (1993), and Bagchi et al. (1990) have compared classes of designs, and examined the implications of recovering and combining information from the other strata. Chang and Notz (1994) also compare classes of designs but only for the fixed effects analysis. The authors cited in this paragraph consider all treatment contrasts and do not look at the case of control vs. test treatment contrasts.

Bailey (1993) compares Semi-Latin squares (row-column designs with nested plots) with the best Incomplete Block Designs. Under the fixed effects model the best IBD gives lower average variance. After recovering inter-block information and combining, a criterion is given in terms of the variances for choosing the 'better' design. Similarly, after recovering further information from the row and column strata and combining, another criterion is given for choosing the better design.

Morgan and Uddin (1993) look at superblocks with nested rows and columns. They show that BNRCs are superior to Balanced Incomplete Block Designs with nested rows and columns (BIBRC) under the fixed effects model. The BIBRC designs are binary in blocks and have a completely symmetric information matrix. After recovering row, column and block strata information and combining, they give a criterion for choosing between a 'Series A' BIBRC and a 'Series A' BNRC (which has rows forming blocks of a balanced incomplete block design). The reader should refer to the paper for the definitions of 'Series A' BIBRC and a 'Series A' BNRC.

Bagchi et al. (1990) also consider superblocks with nested rows and columns. They provide an example of a non-binary design which is better than a binary design under the fixed effects model. Under the mixed effects model they give the criterion for choosing between a binary nested row-column design with completely symmetric C-matrix (BIB-RC) and a BN-RC design.

Following the methods for comparing designs described by Bailey (1993), Morgan and Uddin (1993) and Bagchi et al. (1990), the present paper shows the criterion required for one non-binary design to be better than another non-binary design in terms of giving lower variances for treatment vs. control comparisons in the nested row-column design setup. One design satisfies condition (i) given previously, but neither design satisfies condition (ii).

2. The Designs

Consider a Balanced Incomplete Block Design (BIBD) named Λ, with v treatments (labelled $x = 1, \ldots, v$) arranged in $b = v(v-1)/2$ blocks each of size 2. Now consider a nested row-column design Λ', constructed from Λ. Each square in Λ' is termed a 'superblock':

$$\Lambda' = \left\{ \begin{array}{|c|c|} \hline x & 0 \\ \hline 0 & y \\ \hline \end{array} : \{x, y\} \text{ is a block in } \Lambda \right\} \text{ repeated } c \text{ times,}$$

where

$$c = \begin{cases} 1 & \text{if } v \text{ is odd,} \\ 2 & \text{if } v \text{ is even.} \end{cases}$$

Treatment 0 is the control and x and y are test treatments. There are $cv(v-1)/2$ superblocks. John and Williams (1995) say that a connected design is one that cannot be split up into groups of blocks where treatments in one group are different from treatments in any other group. This design is therefore connected. This implies that every treatment contrast can be estimated from the plot stratum.

Now consider a third design Γ, which is also a row-column design, and is formed as follows:

$$\Gamma = \left\{ \begin{array}{|c|c|} \hline x & 0 \\ \hline 0 & x \\ \hline \end{array} : x = 1, \ldots, v \right\} \text{ repeated } \left(\frac{v-1}{2} \right) c \text{ times.}$$

Γ consists of $cv(v-1)/2$ superblocks and is a connected design.

Note that $r_0^{\Lambda'} = cv(v-1)$ and $r_0^{\Gamma} = cv(v-1)$ are the replications of the control treatment in design Λ' and Γ respectively. Also, $r^{\Lambda'} = c(v-1)$ and $r^{\Gamma} = c(v-1)$ are the replications of the test treatments in design Λ' and Γ, respectively.

Consider the variance of the estimated plot level contrasts $(\tau_0 - \tau_v)$, for designs Λ' and Γ, where τ_i is effect of treatment i. In general,

$$\text{Var}(\tau_0 - \tau_x) = \text{Var}(\tau_0 - \tau_v)$$

(for $x = 1, \ldots, v$) since treatments are all replicated a constant number of times, the concurrence for all test treatment pairs is constant and the concurrence for a test treatment with the control is constant. As is shown in Section 5, design Γ always gives a lower value for $\text{Var}(\tau_0 - \tau_x)$ than design Λ' at plot level. This is unintuitive, since design Γ departs the further from binarity. However, the result is confirmed by what is shown in the papers by Bagchi et al. (1990) and Morgan and Uddin (1993).

The design with lower variance on the $(\tau_0 - \tau_x)$ contrast is said to be the 'better' design, since it provides more accurate estimates of the differences between test treatments and the control. However, after combining information from row and column strata, which is the better design depends on a criterion involving the stratum variances.

2.1. *Contrasts*

The information matrix, $\mathbf{C}$, of a nested row-column design is

$$\mathbf{C} = \mathbf{r}^{\delta} - \frac{\mathbf{N}_p\mathbf{N}_p'}{q} - \frac{\mathbf{N}_q\mathbf{N}_q'}{p} + \frac{\mathbf{N}_b\mathbf{N}_b'}{pq}.$$

In Pearce (1976) notation, $\frac{\mathbf{N}_p\mathbf{N}_p'}{q}$, $\frac{\mathbf{N}_q\mathbf{N}_q'}{p}$ and $\frac{\mathbf{N}_b\mathbf{N}_b'}{pq}$ are the 'weighted concurrence' matrices for rows, columns, and superblocks respectively, such that

$$\mathbf{N}_p = \mathbf{X}'\mathbf{Z}_p,$$

$$\mathbf{N}_q = \mathbf{X}'\mathbf{Z}_q,$$

$$\mathbf{N}_b = \mathbf{X}'\mathbf{Z}_b,$$

where:

- $\mathbf{r}^{\delta}$ is a $((v+1) \times (v+1))$ diagonal matrix containing $(r_0, r_1, \ldots, r_v)$ for a particular design,
- n is the total number of units ($n = bpq$),
- $\mathbf{X}$ is the $(n \times (v+1))$ design matrix for treatments, with (i, j)'th entry equal to 1 if plot i contains treatment j and 0 otherwise,
- $\mathbf{Z}_b$ is the $(n \times b)$ design matrix for superblocks, with (i, j)'th entry equal to 1 if plot i is contained in superblock j and 0 otherwise,
- $\mathbf{Z}_p$ is the $(n \times bp)$ design matrix for rows, with (i, j)'th entry equal to 1 if plot i is contained in row j and 0 otherwise,
- $\mathbf{Z}_q$ is the $(n \times bq)$ design matrix for columns, with (i, j)'th entry equal to 1 if plot i is contained in column j and 0 otherwise,
- p and q are the number of rows and columns in each superblock.

I define contrasts $\omega_1, \ldots, \omega_v$, which are appropriate for designs Λ' and Γ as follows,

$$\begin{aligned}
\omega_1' &= (v, -1, -1, -1, -1, -1, \ldots, -1), \\
\omega_2' &= (0, 1, -1, 0, 0, 0, \ldots, 0), \\
\omega_3' &= (0, 1, 1, -2, 0, 0, \ldots, 0), \\
&\vdots \\
\omega_v' &= (0, 1, 1, 1, \ldots, 1, -(v-1)),
\end{aligned}$$

where $\boldsymbol{\omega}_j$ $(j = 1, \ldots, v)$ is a vector of size $((v+1) \times v)$. These are mutually orthogonal. I shall show in Sections 3 and 4 that each $\boldsymbol{\omega}_j$ is an eigenvector of $\mathbf{Cr}^{-\delta}$ for both design Λ' and Γ, with corresponding eigenvalue ε_j.

Pearce et al. (1974) and Pearce (1983) define basic contrasts of an Incomplete Block Design in terms of eigenvectors of the matrix $\mathbf{Cr}^{-\delta}$. They are estimated independently. This definition can be extended to Nested Row-Column Designs.

Pearce (1983) considers a set of eigenvectors, where $\mathbf{p}_0 = \mathbf{r}^{\frac{\delta}{2}}\mathbf{1}/\sqrt{n}$, $\mathbf{p}_i'\mathbf{p}_j = 0$ and $\mathbf{p}_j'\mathbf{p}_j = 1$ for $i, j = 0, \ldots, v$. These are

$$
\begin{aligned}
\mathbf{p}_1' &= \frac{1}{\sqrt{2v}}(\sqrt{v}, -1, -1, -1, -1, -1, \ldots, -1),\\
\mathbf{p}_2' &= \frac{1}{\sqrt{2}}(0, 1, -1, 0, 0, 0, \ldots, 0),\\
\mathbf{p}_3' &= \frac{1}{\sqrt{6}}(0, 1, 1, -2, 0, 0, \ldots, 0),\\
&\vdots\\
\mathbf{p}_v' &= \frac{1}{\sqrt{v(v-1)}}(0, 1, 1, 1, \ldots, 1, -(v-1)).
\end{aligned}
$$

Pearce (1983) defines the basic contrasts $\mathbf{z}_1, \ldots, \mathbf{z}_v$. These are such that $\mathbf{z}_0 = \mathbf{r}/\sqrt{n}$ and $\mathbf{z}_j = \mathbf{r}^{\frac{\delta}{2}}\mathbf{p}_j$ for $j = 1, \ldots, v$, and they satisfy $\mathbf{z}_i'\mathbf{r}^{-\delta}\mathbf{z}_j = 0$ and $\mathbf{z}_j'\mathbf{r}^{-\delta}\mathbf{z}_j = 1$. Pearce (1983) says that the basic contrasts are independent. The basic contrasts are eigenvectors of $\mathbf{Cr}^{-\delta}$ i.e.

$$\mathbf{Cr}^{-\delta}\mathbf{z}_j = \varepsilon_j \mathbf{z}_j,$$

where ε_j is the canonical efficiency factor corresponding to the basic contrast $\mathbf{z}_j$ in the plot stratum. Pearce *et al.* (1974) also detail these basic contrasts but name them as $\mathbf{c}_1, \ldots, \mathbf{c}_v$.

Since $\mathbf{z}_j = \mathbf{r}^{\frac{\delta}{2}}\mathbf{p}_j$, $r_0^{\Lambda'} = vr^{\Lambda'}$, $r_0^{\Gamma} = vr^{\Gamma}$ and $r^{\Lambda'} = r^{\Gamma}$, if we let $r = r^{\Lambda'}$, then

$$
\begin{aligned}
\mathbf{z}_1' &= \sqrt{\frac{r}{2v}}(v, -1, -1, -1, -1, -1, \ldots, -1),\\
\mathbf{z}_2' &= \sqrt{\frac{r}{2}}(0, 1, -1, 0, 0, 0, \ldots, 0),\\
\mathbf{z}_3' &= \sqrt{\frac{r}{6}}(0, 1, 1, -2, 0, 0, \ldots, 0),\\
&\vdots\\
\mathbf{z}_v' &= \sqrt{\frac{r}{v(v-1)}}(0, 1, 1, 1, \ldots, 1, -(v-1)),
\end{aligned}
$$

Since $\boldsymbol{\omega}_j$ is a multiple of $\mathbf{z}_j$ it can be seen that

$$\mathbf{C}\mathbf{r}^{-\delta}\boldsymbol{\omega}_j = \varepsilon_j\boldsymbol{\omega}_j, \quad (1)$$

where ε_j is the canonical efficiency factor corresponding to contrast $\boldsymbol{\omega}_j$, in the plot stratum. The contrasts $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_j$ are estimated independently if $i \neq j$, because they are multiples of the basic contrasts. To simplify things it is easier to work with eigenvectors $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_v$.

See Nelder (1965) and James and Wilkinson (1971) for a discussion of canonical efficiency factors. Nelder (1965) finds the canonical efficiency factors in all strata as eigenvalues of the information matrix. He does not, however, name these eigenvalues as canonical efficiency factors. James and Wilkinson (1971) give details of the canonical decomposition method. They state that 'canonical efficiency factors measure the extent of nonorthogonality'.

Most authors refer to canonical efficiency factors specifically as those found for the plot stratum; they are thought of as the amount of information available at plot level from a contrast. However, canonical efficiency factors can be found for a contrast in the row, column, and superblock strata. Houtman and Speed (1983) define general balance as the concept that the information matrices for the designs formed from the rows, columns and superblocks, are spanned by the same eigenvectors. These matrices are of the forms

$$\mathbf{C}_p = \mathbf{r}^\delta - \frac{\mathbf{N}_p\mathbf{N}_p'}{q},$$

$$\mathbf{C}_q = \mathbf{r}^\delta - \frac{\mathbf{N}_q\mathbf{N}_q'}{p}$$

and

$$\mathbf{C}_b = \mathbf{r}^\delta - \frac{\mathbf{N}_b\mathbf{N}_b'}{pq},$$

respectively.

Designs Λ' and Γ are generally balanced and have the same basic contrasts. So, the contrast $\boldsymbol{\omega}_j$ is an eigenvector of some of the component parts of the $\mathbf{C}$-matrix (namely the weighted concurrence matrices) postmultiplied by $\mathbf{r}^{-\delta}$. This gives the eigenvalues corresponding to the contrast $\boldsymbol{\omega}_j$ in the row, column, and superblock strata respectively. That is, $\boldsymbol{\omega}_j$ is an eigenvector of

$$\left(\frac{\mathbf{N}_p\mathbf{N}_p'}{q}\right)\mathbf{r}^{-\delta},$$

which gives the eigenvalue for the contrast in the row stratum. Contrast $\boldsymbol{\omega}_j$ is also an eigenvector of

$$\left(\frac{\mathbf{N}_q\mathbf{N}_q'}{p}\right)\mathbf{r}^{-\delta},$$

which gives the eigenvalue for the contrast in the column stratum. Contrast $\boldsymbol{\omega}_j$ is also an eigenvector of

$$\left(\frac{\mathbf{N}_b\mathbf{N}_b'}{pq}\right)\mathbf{r}^{-\delta},$$

which gives the eigenvalue for the contrast in the superblock stratum. The eigenvalues are the same for $\boldsymbol{\omega}_j$ as for the corresponding basic contrast. The canonical efficiency factor, CEF_{Sj}, for a contrast $\boldsymbol{\omega}_j$ in stratum S, for a nested row-column design, is given by

$$\begin{aligned} CEF_{SBj} &= EV_{SBj}, \\ CEF_{Rj} &= EV_{Rj} - CEF_{SBj}, \\ CEF_{Cj} &= EV_{Cj} - CEF_{SBj}, \\ CEF_{Pj} &= EV_{Pj} - CEF_{Cj} - CEF_{Rj} - CEF_{SBj}, \end{aligned}$$

where $S = P, C, R, SB$ represent the plot, column, row and superblock strata, respectively. EV_{Sj} is the eigenvalue of the weighted concurrence matrix post-multiplied by $\mathbf{r}^{-\delta}$ for contrast $\boldsymbol{\omega}_j$ in strata $S = C, R, SB$, and is the eigenvalue of the information matrix post-multiplied by $\mathbf{r}^{-\delta}$ for contrast $\boldsymbol{\omega}_j$ in the plot stratum.

The canonical efficiency factors for stratum S are the same for contrasts $\boldsymbol{\omega}_2, \ldots, \boldsymbol{\omega}_v$ since the design is formed from a BIBD. Since less effort is needed to compute the canonical efficiency factors for the $\boldsymbol{\omega}_2$ contrast than any of the remaining $\boldsymbol{\omega}_3, \ldots, \boldsymbol{\omega}_v$ contrasts, it is simpler to calculate canonical efficiency factors for $\boldsymbol{\omega}_2$ only.

3. Statistical properties of design Λ'

Examine superblocks in which treatments 1 and 2 occur (either together or separately):

$$\Lambda' = \left\{ \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 2 \\ \hline \end{array} \; \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & y \\ \hline \end{array} \; \begin{array}{|c|c|}\hline 2 & 0 \\ \hline 0 & y \\ \hline \end{array} \ldots \right\} \text{ repeated } c \text{ times with } y = 3, \ldots, v.$$

The eigenvalues and canonical efficiency factor tables for the design Λ' are in Table 1 and Table 2.

Contrast $\boldsymbol{\omega}_1$ is orthogonal to superblocks, giving an eigenvalue of 0. However, averaging the $\boldsymbol{\omega}_2$ contrast over the superblocks, then averaging over treatments, gives an eigenvalue corresponding to $\boldsymbol{\omega}_2$. Contrast $\boldsymbol{\omega}_2$ represents $(\tau_1 - \tau_2)$, and is orthogonal

Table 1. Eigenvalues for design Λ'

Stratum	ω_1	ω_2	...	ω_v
Superblock	0	$\frac{1}{4}(\frac{v-2}{v-1})$	...	$\frac{1}{4}(\frac{v-2}{v-1})$
Rows	0	$\frac{1}{2}$	...	$\frac{1}{2}$
Columns	0	$\frac{1}{2}$	...	$\frac{1}{2}$
Plots	1	1	...	1

Table 2. Canonical Efficiency Factors for design Λ'

Stratum	ω_1	ω_2	...	ω_v
Superblock	0	$\frac{1}{4}(\frac{v-2}{v-1})$	...	$\frac{1}{4}(\frac{v-2}{v-1})$
Rows	0	$\frac{1}{4}(\frac{v}{v-1})$	...	$\frac{1}{4}(\frac{v}{v-1})$
Columns	0	$\frac{1}{4}(\frac{v}{v-1})$	...	$\frac{1}{4}(\frac{v}{v-1})$
Plots	1	$\frac{1}{4}(\frac{v-2}{v-1})$	...	$\frac{1}{4}(\frac{v-2}{v-1})$

to the c superblocks in which treatments 1 and 2 occur together, but is not orthogonal to the $c(v-2)$ superblocks containing treatment 1 and the $c(v-2)$ superblocks containing treatment 2. The eigenvalue for the ω_2 contrast is the same for contrasts $\omega_3, \ldots, \omega_v$, so the eigenvalue entry for contrast ω_k in the superblock stratum is

$$EV_{SBk} = \frac{1}{4}\left(\frac{v-2}{v-1}\right) \text{ for } k = 2, \ldots, v.$$

The contrast ω_1 is orthogonal to rows giving an eigenvalue of 0. However, the ω_2 contrast is not orthogonal to rows containing treatment 1 and those rows containing treatment 2. By averaging the contrast over rows and treatments, the eigenvalue is found. Swapping the role of rows and columns gives the same design. So

$$EV_{Sk} = \frac{1}{2} \text{ for } k = 2, \ldots, v \text{ and } S = R, C.$$

Recall that $\boldsymbol{\tau}$ is the vector of treatment effects, and that if $CEF_{Sj} \neq 0$ then there is a least squares estimator of $\boldsymbol{\omega}_j'\boldsymbol{\tau}$ using information from stratum S. We write this estimator as $(\boldsymbol{\omega}_j)_S$ and its variance as $\mathrm{Var}(\boldsymbol{\omega}_j)_S$. The variance for contrast $\boldsymbol{\omega}_j'\boldsymbol{\tau}$ in stratum S is found as

$$\text{Var}(\boldsymbol{\omega}_j)_S = \left(\frac{1}{CEF_{Sj}} \sum_i \frac{\omega_{ji}^2}{r_i} \right) \xi_S, \qquad (2)$$

where r_i is the replication of treatment i for $i = 0, \ldots, v$, and ξ_S is the particular stratum variance. Thus the stratum variances in a nested row-column design are

- ξ_P – the plot stratum variance,
- ξ_R – the row stratum variance,
- ξ_C – the column stratum variance.

The variance of a control vs. treatment contrast is

$$\text{Var}(\tau_0 - \tau_x)_S = \frac{1}{v^2}(\text{Var}(\boldsymbol{\omega}_1)_S + \text{Var}(\boldsymbol{\omega}_v)_S). \qquad (3)$$

A commonly used method to combine information from estimates made in two different strata is

$$(\boldsymbol{\omega}_j)_{SU} = \left(\frac{\frac{(\boldsymbol{\omega}_j)_S}{\text{Var}(\boldsymbol{\omega}_j)_S} + \frac{(\boldsymbol{\omega}_j)_U}{\text{Var}(\boldsymbol{\omega}_j)_U}}{\frac{1}{\text{Var}(\boldsymbol{\omega}_j)_S} + \frac{1}{\text{Var}(\boldsymbol{\omega}_j)_U}} \right)$$

with corresponding variance

$$\text{Var}(\boldsymbol{\omega}_j)_{SU} = \frac{1}{\frac{1}{\text{Var}(\boldsymbol{\omega}_j)_S} + \frac{1}{\text{Var}(\boldsymbol{\omega}_j)_U}}, \qquad (4)$$

where S, U and SU represent strata S, U and the combined stratum SU, respectively.

Using equations (2) and (3), at plot level

$$\text{Var}(\tau_0 - \tau_v)_P = \frac{1}{v^2} \left(\frac{2v}{c(v-1)} + \frac{4v(v-1)}{c(v-2)} \right) \xi_P. \qquad (5)$$

At row level, there is no information on the $\boldsymbol{\omega}_1$ contrast, hence the combined plot and row level information for $\text{Var}(\boldsymbol{\omega}_1)$ remains the same as that for plot level information. Combined plot and row information for $\text{Var}(\boldsymbol{\omega}_v)$ is found using equation (4). The combined plot and row information for $\text{Var}(\tau_0 - \tau_x)$ on the Λ' design can thus be shown to be

$$\text{Var}(\tau_0 - \tau_x)_{PR} = \frac{1}{v^2} \left(\frac{2v}{c(v-1)} + \frac{4v(v-1)\xi_R}{c(v-2)\xi_R + cv\xi_P} \right) \xi_P. \qquad (6)$$

The combined plot and column information on contrasts $\text{Var}(\tau_0 - \tau_x)$ in the Λ'

design is

$$\mathrm{Var}(\tau_0 - \tau_x)_{PC} = \frac{1}{v^2}\left(\frac{2v}{c(v-1)} + \frac{4v(v-1)\xi_C}{c(v-2)\xi_C + cv\xi_P}\right)\xi_P. \tag{7}$$

The combined plot, row and column information for $\mathrm{Var}(\tau_0 - \tau_x)$ in the Λ' design is

$$\mathrm{Var}(\tau_0 - \tau_x)_{PRC} = \frac{1}{v^2}\left(\frac{2v}{c(v-1)} + \frac{4v(v-1)\xi_R\xi_C}{c(v-2)\xi_R\xi_C + cv\xi_P\xi_C + cv\xi_P\xi_R}\right)\xi_P. \tag{8}$$

4. Statistical properties of design Γ

In general design Γ is of the form

$$\Gamma = \left\{ \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \; \begin{array}{|c|c|}\hline 2 & 0 \\ \hline 0 & 2 \\ \hline \end{array} \; \begin{array}{|c|c|}\hline y & 0 \\ \hline 0 & y \\ \hline \end{array} \cdots \right\} \text{ repeated } c \text{ times with } y = 3,\ldots,v.$$

The eigenvalue and canonical efficiency factor tables for the design Γ are shown in Tables 3 and 4.

Table 3. Eigenvalues for design Γ

Stratum	ω_1	ω_2	...	ω_v
Superblock	0	$\frac{1}{2}$	...	$\frac{1}{2}$
Rows	0	$\frac{1}{2}$	...	$\frac{1}{2}$
Columns	0	$\frac{1}{2}$	...	$\frac{1}{2}$
Plots	1	1	...	1

Table 4. Canonical Efficiency Factors for design Γ

Stratum	ω_1	ω_2	...	ω_v
Superblock	0	$\frac{1}{2}$	...	$\frac{1}{2}$
Rows	0	0	...	0
Columns	0	0	...	0
Plots	1	$\frac{1}{2}$	...	$\frac{1}{2}$

Contrast ω_1 on superblocks is easily seen to be orthogonal to all superblocks giving an eigenvalue of 0. However, contrast ω_2 is not orthogonal to those superblocks containing test treatment 1 and those superblocks containing test treatment 2. So averaging over superblocks and treatments gives

$$EV_{SBk} = \frac{1}{2} \text{ for } k = 2, \ldots, v.$$

The contrast ω_1 is orthogonal to rows giving an eigenvalue of 0. Note that contrast ω_2 is not orthogonal to those rows containing test treatment 1 or those rows containing test treatment 2 and it can easily be seen that for ω_2 the row eigenvalue in this design is the same as in the design Λ' at row level. Therefore the appropriate eigenvalue in design Γ is as above, and similarly the same result applies for columns, since rows and columns can be interchanged to give the same design. Therefore

$$EV_{Sk} = \frac{1}{2} \text{ for } k = 2, \ldots, v \text{ and } S = R, C.$$

Using equations (2), and (3),

$$\text{Var}(\tau_0 - \tau_x)_P = \frac{1}{v^2}\left(\frac{2v}{c(v-1)} + \frac{2v}{c}\right)\xi_P. \tag{9}$$

Information does not exist on either the ω_1 and ω_v contrast, at either row or column level. In turn, this results in the combined plot and row information, plot and column information, or plot, row and column information for $\text{Var}(\tau_0 - \tau_x)$ being the same as the plot value, given in equation (9).

5. Comparing Λ' and Γ

When comparing two designs, the 'most binary' design is expected to give the lower variance. However, Bagchi et al. (1990) have shown that this is not always the case. Comparing values for $\text{Var}(\tau_0 - \tau_x)$ at plot level, we use equations (5) and (9). Design Λ' has a greater value for $\text{Var}(\tau_0 - \tau_x)_P$ than design Γ, if and only if

$$\frac{1}{v^2}\left(\frac{2v}{c(v-1)} + \frac{4v(v-1)}{c(v-2)}\right)\xi_P > \frac{1}{v^2}\left(\frac{2v}{c(v-1)} + \frac{2v}{c}\right)\xi_P.$$

This gives

$$v > 0.$$

Since this must always be true, the design Γ must always provide the lower variance for $(\tau_0 - \tau_x)_P$ and so is the better design.

Designs Λ' and Γ can also be compared after combining plot and row information (using equations 6 and 9). The value of $\mathrm{Var}(\tau_0 - \tau_x)_{PR}$ is greater for design Λ' than for design Γ, if and only if

$$\frac{1}{v^2}\left(\frac{2v}{c(v-1)} + \frac{4v(v-1)\xi_R}{c(v-2)\xi_R + cv\xi_P}\right)\xi_P > \frac{1}{v^2}\left(\frac{2v}{c(v-1)} + \frac{2v}{c}\right)\xi_P.$$

This gives

$$1 > \frac{\xi_P}{\xi_R}.$$

Since for 'good rows' this must be true, design Γ must always provide the lower variance for $(\tau_0 - \tau_x)_{PR}$ and so is the better design.

Comparing Λ' and Γ after combining plots, rows and columns (using equations 8 and 9), the 'better' design for $(\tau_0 - \tau_x)_{PRC}$, is dependent on the following criterion,

$$\frac{1}{v^2}\left(\frac{2v}{c(v-1)} + \frac{4v(v-1)\xi_R\xi_C}{c(v-2)\xi_R\xi_C + cv\xi_P\xi_R + cv\xi_P\xi_C}\right)\xi_P > \frac{1}{v^2}\left(\frac{2v}{c(v-1)} + \frac{2v}{c}\right)\xi_P.$$

This gives

$$1 > \frac{\xi_P}{\xi_R} + \frac{\xi_P}{\xi_C}.$$

For 'good rows' $\frac{\xi_P}{\xi_R} < 1$ and for 'good columns' $\frac{\xi_P}{\xi_C} < 1$. Therefore,

- Γ is the better design if $0 < \frac{\xi_P}{\xi_R} + \frac{\xi_P}{\xi_C} < 1$,
- Λ' is the better design if $1 < \frac{\xi_P}{\xi_R} + \frac{\xi_P}{\xi_C} < 2$.

The value of $\mathrm{Var}(\omega_1)_P$ is the same for both designs. The design giving smaller value of $\mathrm{Var}(\omega_v)_P$, and larger value for the corresponding canonical efficiency factor, is the better design. Tables 2 and 4 show that the value of $\mathrm{Var}(\omega_v)_P$ is larger in design Λ'. Consequently, the value of $\mathrm{Var}(\tau_0 - \tau_x)_P$ is larger in Λ', implying that Γ is a better design.

At combined plot and row, plot and column, or plot, row and column in Λ', the value for $\mathrm{Var}(\omega_1)$ is the same as the plot value. In the combining process, the value of $\mathrm{Var}(\omega_v)$ decreases in comparison to the plot level value, and hence the value of $\mathrm{Var}(\tau_0 - \tau_x)$ decreases. In design Γ at combined plot and row, plot and column, or plot, row and column any information on $\mathrm{Var}(\omega_1)$, $\mathrm{Var}(\omega_v)$ and $\mathrm{Var}(\tau_0 - \tau_x)$ equals that achieved at plot level only. It is found, at combined plot and row level, that the Γ design is always better. The same argument holds for combined plot and column information. At combined plot, row and column level, the best design depends on $\frac{\xi_P}{\xi_R} + \frac{\xi_P}{\xi_C} < 1$ given that $\frac{\xi_P}{\xi_R} < 1$ and $\frac{\xi_P}{\xi_C} < 1$. This means that the better design is

– Γ if $0 < \frac{\xi_P}{\xi_R} + \frac{\xi_P}{\xi_C} < 1$,
– Λ' if $1 < \frac{\xi_P}{\xi_R} + \frac{\xi_P}{\xi_C} < 2$.

Often ξ_P, ξ_R and ξ_C are unknown prior to an experiment. However, previous experiments of similar sizes may be used to give an idea of the sizes of the stratum variances, thus allowing a more informed choice of design, particularly if intending to recover and combine row and column information.

6. Example

Kachlicka and Mejza (1995) give an example of a 2×2 nested row-column design with 3 sub-plots per whole-plot. The potential experiment would observe potato crop yields under the influence of three levels of irrigation (on whole plots) and three levels of nitrogen (on sub-plots). Following on from their example and simplifying, we consider a potential experiment where potato yield is observed only under the influence of three levels of irrigation. The control treatment is absence of irrigation.

Consider the following Balanced Incomplete Block Design, Λ, with $v = 3$, $b = 3$, $r = 2$, and block size 2. Call this particular Balanced Incomplete Block Design Λ_1.

$$\Lambda_1 = \begin{array}{|c|}\hline 1\,2 \\ \hline 2\,3 \\ \hline 3\,1 \\ \hline \end{array}$$

Design Λ_1 leads to the construction of Λ_1' as below. The design is repeated once since $v = 3$ is odd.

$$\Lambda_1' = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 2 & 0 \\ \hline 0 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 3 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$$

The contrasts are

$$\begin{aligned} \omega_1' &= [3, -1, -1, -1], \\ \omega_2' &= [0, 1, -1, 0], \\ \omega_3' &= [0, 1, 1, -2]. \end{aligned}$$

The variances are summarized in Table 5.

There is no information on contrast ω_1 in either the row or the column strata, so when combining information from other strata, no extra information is achieved and the value for $\mathrm{Var}(\omega_1)$ is just the value found at plot level. Combining plot and row information; or plot and column information; or plot, row and column information for contrast ω_3 leads to a decrease in the value for $\mathrm{Var}(\omega_3)$. This has the overall effect

Table 5. Variances for design Λ_1'

Stratum	$\mathrm{Var}(\omega_1)$	$\mathrm{Var}(\omega_3)$	$\mathrm{Var}(\tau_0-\tau_3)$
Plot	$3\xi_P$	$24\xi_P$	$3\xi_P$
Row	∞	$8\xi_R$	∞
Column	∞	$8\xi_C$	∞
Plot & Row	$3\xi_P$	$\left(\frac{24\xi_P\xi_R}{\xi_R+3\xi_P}\right)$	$\left(\frac{3\xi_R+\xi_P}{\xi_R+3\xi_P}\right)\xi_P$
Plot & Column	$3\xi_P$	$\left(\frac{24\xi_P\xi_C}{\xi_C+3\xi_P}\right)$	$\left(\frac{3\xi_C+\xi_P}{\xi_C+3\xi_P}\right)\xi_P$
Plot, Row & Column	$3\xi_P$	$\left(\frac{24\xi_P\xi_C\xi_R}{\xi_R\xi_C+3\xi_P\xi_C+3\xi_P\xi_R}\right)$	$\left(\frac{3\xi_R\xi_C+\xi_P\xi_C+\xi_P\xi_R}{\xi_R\xi_C+3\xi_P\xi_C+3\xi_P\xi_R}\right)\xi_P$

of decreasing the combined value of $\mathrm{Var}(\tau_0-\tau_3)$ compared to the plot level value $\mathrm{Var}(\tau_0-\tau_3)$.

Since $c=1$, Γ_1 is

$$\Gamma_1 = \boxed{\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array}} \quad \boxed{\begin{array}{|c|c|}\hline 2 & 0 \\ \hline 0 & 2 \\ \hline\end{array}} \quad \boxed{\begin{array}{|c|c|}\hline 3 & 0 \\ \hline 0 & 3 \\ \hline\end{array}}.$$

Table 6 summarizes the variances.

No information exists for either contrast ω_1 or ω_3 other than at plot level, so no extra information is gained by combining. When comparing at plot level, since

$$3\xi_P > 1\xi_P$$

then $\mathrm{Var}(\tau_0-\tau_x)_P$ is greater for Λ_1' than for Γ_1, concluding that Γ_1 is the better of the two designs at plot level.

Table 6. Variances for design Γ_1

Stratum	$\mathrm{Var}(\omega_1)$	$\mathrm{Var}(\omega_3)$	$\mathrm{Var}(\tau_0-\tau_3)$
Plot	$3\xi_P$	$6\xi_P$	$1\xi_P$
Row	∞	∞	∞
Column	∞	∞	∞
Plot & Row	$3\xi_P$	$6\xi_P$	$1\xi_P$
Plot & Column	$3\xi_P$	$6\xi_P$	$1\xi_P$
Plot, Row & Column	$3\xi_P$	$6\xi_P$	$1\xi_P$

$\mathrm{Var}(\tau_0 - \tau_x)_{PR}$ is greater in design Λ_1' than in Γ_1 if and only if

$$\xi_P \left(\frac{3\xi_R + \xi_P}{\xi_R + 3\xi_P} \right) > \xi_P.$$

Rearranging, this gives $1 > \xi_P/\xi_R$ and since realistically ξ_P/ξ_R should be less than 1, we see that Γ_1 is the better design if $0 < \xi_P/\xi_R < 1$. Swapping the role of rows and columns, the same design is obtained (simply substitute ξ_C for ξ_R above, where ξ_C is the column variance).

The design Λ_1' has greater variance than Γ_1 at combined plot, row and column level if and only if

$$\xi_P \left(\frac{3\xi_R\xi_C + \xi_P\xi_R + \xi_P\xi_C}{\xi_R\xi_C + 3\xi_P\xi_R + 3\xi_P\xi_C} \right) > \xi_P.$$

After rearranging, this gives $1 > \xi_P/\xi_C + \xi_P/\xi_R$. But, realistically, $\xi_P/\xi_R < 1$ and $\xi_P/\xi_C < 1$, so the better design is

- Γ_1 if $0 < \frac{\xi_P}{\xi_C} + \frac{\xi_P}{\xi_R} < 1$ subject to $\xi_P/\xi_R < 1$ and $\xi_P/\xi_C < 1$ being satisfied,
- Λ_1' if $1 < \frac{\xi_P}{\xi_C} + \frac{\xi_P}{\xi_R} < 2$ subject to $\xi_P/\xi_R < 1$ and $\xi_P/\xi_C < 1$ being satisfied.

To summarize, Γ_1 is the better design for comparing no irrigation with the three levels of irrigation at plot level. Design Γ_1 is the better design for comparing no irrigation with the three levels of irrigation at combined plot and row level or at combined plot and column level. There is a choice of which is the better design for comparing no irrigation with the three levels of irrigation at combined plot, row and column level. This is dependant on the ratio of the stratum variances.

7. Conclusion

In the Λ_1' design, the ω_1 contrast is orthogonal to all the superblocks, rows and columns and as a result these strata provide no information on this contrast other than from the plot stratum. The remaining contrasts, $\omega_2, \ldots, \omega_v$, are not orthogonal to all superblocks, all rows or all columns, and this results in non-zero entries in the canonical efficiency factor table. For these contrasts, the row and column strata will contain the most information (shown by larger canonical efficiency factors).

In the Γ design the ω_1 contrast does not provide information on contrasts in the superblock, row and column strata. All the information is from the plot level comparisons.

Two non-binary designs are compared. Under the fixed effects model design Γ is the better for estimating $(\tau_0 - \tau_x)$ contrasts. In practice design Γ is still the better after recovery and combination of plot and row, or plot and column information. However, after combination of plot, row and column information the situation depends on the ratios of the stratum variances.

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Porównanie dwóch zagnieżdżonych układów wierszowo-kolumnowych zawierających obiekt kontrolny

Streszczenie

W pracy rozważane są zagnieżdżone, wierszowo-kolumnowe układy doświadczalne o superblokach zawierających dwa wiersze i dwie kolumny. Dwie jednostki w każdym bloku przeznaczone są na obiekt kontrolny, który występuje raz w każdym wierszu i raz w każdej kolumnie. Takie rozwiązanie umożliwia rozmieszczenie obiektów badanych na wiele różnych sposobów. Praca poświęcona jest dwu układom które nie są binarne, lecz które mają te same wektory własne jako kontrasty bazowe. Przedmiotem zainteresowania są kontrasty pomiędzy badanymi obiektami a kontrolą. Układy są porównywane pod względem analizy efektów stałych, a następnie pod względem odzyskiwania informacji z różnych warstw. Pokazano, że gdy odzyska się i połączy informację z poziomu poletek, wierszy i kolumn, postać układu posiadającego mniejszą wariancję porównań typu obiekt-kontrola zależy od wariancji poletkowych, wierszowych i kolumnowych.

Słowa kluczowe: kanoniczne współczynniki efektywności, kombinowanie informacji, obiekt kontrolny, planowanie doświadczeń, zagnieżdżone układy wierszowo-kolumnowe